SPECTRAL THEORY, HAUSDORFF DIMENSION AND THE TOPOLOGY OF HYPERBOLIC 3-MANIFOLDS

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Abstract. Let M be a compact 3-manifold whose interior admits a complete hyperbolic structure. We let $\Lambda(M)$ be the supremum of $\lambda_0(N)$ where N varies over all hyperbolic 3-manifolds homeomorphic to the interior of M. Similarly, we let $D(M)$ be the infimum of the Hausdorff dimensions of limit sets of Kleinian groups whose quotients are homeomorphic to the interior of M. We observe that $\Lambda(M) = D(M)(2 - D(M))$ if M is not handlebody or a thickened torus. We characterize exactly when $\Lambda(M) = 1$ and $D(M) = 1$ in terms of the characteristic submanifold of the incompressible core of M.

1. Introduction

When a closed 3-manifold admits a hyperbolic structure, this structure is unique by Mostow's rigidity theorem [26]. It follows that any invariant of the hyperbolic structure is automatically a topological invariant. One example of this is the hyperbolic volume, which agrees with Gromov's simplicial norm (see [15]).

In this paper we will consider geometrically derived invariants for compact 3-manifolds with boundary whose interiors admit complete hyperbolic metrics of infinite volume. In this case the hyperbolic structure is not unique, and in fact, Thurston's geometrization theorem together with the Ahlfors-Bers quasiconformal deformation theory guarantee that there is at least a one complex dimensional space of hyperbolic structures.

To obtain a topological invariant in this context, one may begin with a natural geometric invariant of a hyperbolic metric, and minimize (or maximize) it over the class of all hyperbolic metrics on a given 3-manifold. In particular, given a hyperbolic 3-manifold $N = \mathbf{H}^3/\Gamma$ we will consider the bottom $\lambda_0(N)$ of the L^2 spectrum of the Laplacian,

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and the Hausdorff dimension $d(N)$ of the limit set L_Γ of Γ. (See section 2 for more precise definitions). Although these two geometric invariants seem very different, the work of Patterson, Sullivan and others has established that they are closely related.

Call a compact, orientable 3-manifold M *hyperbolizable* if its interior admits a complete hyperbolic metric. This is a topological condition: Thurston's geometrization theorem asserts that an orientable compact 3-manifold with non-empty boundary is hyperbolizable if and only if it is irreducible and atoroidal. Given a hyperbolizable M, we will define

$$\Lambda(M) = \sup \lambda_0(N)$$

and

$$D(M) = \inf d(N),$$

where N varies over the space of all hyperbolic 3-manifolds homeomorphic to the interior of M.

Very roughly speaking, we will find that D increases with topological complexity, and Λ decreases. The purpose of this paper is to establish some quantitative aspects of this intuition. In particular we will characterize exactly which manifolds have $\Lambda(M) = 1$, the maximal possible value.

Let us introduce some topological notation. A compact irreducible manifold M has an *incompressible core*, which is a (possibly disconnected) submanifold with incompressible boundary, from which M is obtained by adding 1-handles. A compact irreducible 3-manifold M with incompressible boundary is called a *generalized book of I-bundles* if one may find a disjoint collection A of essential annuli in M such that each component R of the manifold obtained by cutting M along A is either a solid torus, a thickened torus, or homeomorphic to an I-bundle such that $\partial R \cap \partial M$ is the associated ∂I-bundle.

The following is a "scorecard" for the basic facts about Λ and D for hyperbolizable manifolds.

<table>
<tr><th>M (hyperbolizable)</th><th>Λ</th><th>D</th><th>Relation</th></tr>
<tr><td>Handlebody or thickened torus</td><td>1</td><td>0</td><td></td></tr>
<tr><td>Not solid or thickened torus, but ∂M is a union of tori</td><td>0</td><td>2</td><td rowspan="3">$\Lambda = D(2-D)$</td></tr>
<tr><td>Incompressible core consists of generalized books of I-bundles</td><td>1</td><td>1</td></tr>
<tr><td>Any other manifold with boundary</td><td>$0 < \Lambda < 1$</td><td>$1 < D < 2$</td></tr>
</table>

Our main theorems fill in the first two entries in the last two rows. The rest of the entries follow from known results, as explained in section 2.

Main Theorem I: *Let M be a compact, orientable, hyperbolizable 3-manifold. $\Lambda(M) = 1$ if and only if every component of the incompressible core of M is a generalized book of I-bundles. Otherwise, $\Lambda(M) < 1$.*

Main Theorem II: *Let M be a compact, orientable, hyperbolizable 3-manifold which is not a handlebody or a thickened torus. $D(M) = 1$ if and only if every component of its incompressible core is a generalized book of I-bundles. Otherwise, $D(M) > 1$.*

One could think of these results as analogues of the fact that the Gromov norm of a closed, irreducible 3-manifold M is zero if and only if there exists a collection T of incompressible tori in M such that each component of $M - T$ is a Seifert fibre space. In particular, if M has incompressible boundary, then $D(M) = 1$ if and only if the Gromov norm of the double of M is 0.

2. Preliminaries

In this section we will more carefully define our invariants, derive their basic properties and summarize the proof of the main theorems.

2.1. **Definitions.** If N is a complete orientable hyperbolic 3-manifold, then it is isometric to the quotient of hyperbolic 3-space $\mathbf{H}^3$ by a group Γ of orientation-preserving isometries. Any orientation-preserving isometry extends continuously to a conformal transformation of the sphere at infinity S^2_∞ of hyperbolic 3-space. We define the *domain of discontinuity* Ω_Γ to be the maximal open subset of S^2_∞ on which Γ acts discontinuously. The *limit set* L_Γ of Γ is $S^2_\infty - \Omega_\Gamma$. We define $d(N)$ to be the Hausdorff dimension of the limit set L_Γ of Γ.

Define $\lambda_0(N)$ to be the largest value of λ for which there exists a positive C^∞ function f on N such that $\Delta f + \lambda f = 0$. (Here $\Delta = \operatorname{div} \circ \operatorname{grad}$ denotes the Laplacian. See Sullivan [30] for a discussion of why this is equivalent to other definitions of $\lambda_0(N)$.) Note that $\lambda_0 \geq 0$.

It is clear that if $\widetilde{N}$ covers N then $\lambda_0(N) \leq \lambda_0(\widetilde{N})$. In particular,

$$\lambda_0(N) \leq \lambda_0(\mathbf{H}^3) = 1$$

for any hyperbolic 3-manifold N.

If M is a compact, orientable, hyperbolizable 3-manifold then we let $TT(M)$ denote the set of complete hyperbolic 3-manifolds homeomorphic to the interior of M (TT stands for "topologically tame"; see below). Our invariants can now be written as:

$$\Lambda(M) = \sup\{\lambda_0(N) \mid N \in TT(M)\}$$

and
$$D(M) = \inf\{d(N) \mid N \in TT(M)\}.$$

We will say that an element $N = \mathbf{H}^3/\Gamma$ of $TT(M)$ is *geometrically finite* if there exists a finite collection P of incompressible annuli and tori in ∂M such that $\hat{N} = (\mathbf{H}^3 \cup \Omega_\Gamma)/\Gamma$ is homeomorphic to $M - P$.

We say that a hyperbolic 3-manifold is *topologically tame* if it is homeomorphic to the interior of a compact 3-manifold. This holds by definition for each hyperbolic 3-manifold in $TT(M)$ and we remark that it is conjectured that every hyperbolic 3-manifold with finitely generated fundamental group is topologically tame.

2.2. **Basic properties of Λ and D.** Remarkably, λ_0 and the Hausdorff dimension of the limit set are intimately related. The following theorem, which records that relationship, is due to Sullivan [29] in the case that N is geometrically finite. If N is topologically tame but not geometrically finite, then Canary [11] proved that $\lambda_0(N) = 0$, while Bishop and Jones [4] proved that if N has finitely generated fundamental group and is not geometrically finite then $d(N) = 2$.

Theorem 2.1. *If N is a topologically tame hyperbolic 3-manifold, then $\lambda_0(N) = d(N)(2 - d(N))$ unless $d(N) < 1$, in which case $\lambda_0(N) = 1$.*

Thus λ_0 completely determines d unless $d < 1$. The situation when $d \leq 1$ was analyzed by Sullivan [28] and Braam [8] in the case where N is convex cocompact, by Canary and Taylor [13] when N is geometrically finite, and for all hyperbolic 3-manifolds with finitely generated fundamental groups by Bishop and Jones [4]:

Theorem 2.2. *Let M be a compact, orientable, hyperbolizable 3-manifold. If $N \in TT(M)$ and $d(N) < 1$, then M is a handlebody or a thickened torus. If $d(N) = 1$, then M is either a handlebody or an I-bundle.*

The above two theorems assure us that $D(M)$ and $\Lambda(M)$ are essentially the same invariant:

Corollary 2.3. *If M is a compact, orientable, hyperbolizable 3-manifold, then $\Lambda(M) = D(M)(2 - D(M))$ unless M is a handlebody or a thickened torus.*

Theorem 2.2 also has the following consequence:

Corollary 2.4. *If M is a compact, orientable, hyperbolizable 3-manifold which is not a handlebody or a thickened torus, then $D(M) \geq 1$.*

We note that it follows from work of Beardon [2] that if M is a handlebody, then $D(M) = 0$ and therefore $\Lambda(M) = 1$. If M is a

thickened torus and $N = \mathbf{H}^3/\Gamma \in TT(M)$, then L_Γ is a single point, so again $D(M) = 0$ and $\Lambda(M) = 1$. This completes the first row of the scorecard.

The next proposition completes the second row:

Proposition 2.5. *Let M be a compact, orientable, hyperbolizable 3-manifold which is not a solid torus or a thickened torus. Then $D(M) = 2$ if and only if any boundary component of M is a torus.*

Proof. We first suppose that any boundary component of M is toroidal. This implies that if $N \in TT(M)$ then $N = \mathbf{H}^3/\Gamma$ has finite volume (see Proposition D.3.18 in [3]). In this case $L_\Gamma = S^2_\infty$, which implies that $d(N) = 2$ and hence $D(M) = 2$.

We now suppose that M has a non-toroidal boundary component. In this case, N has infinite volume for any $N \in TT(M)$. Thurston's geometrization theorem (see [24]) guarantees there is a geometrically finite manifold N in $TT(M)$. Sullivan [29] and Tukia [37] proved that if N is geometrically finite and has infinite volume, then $d(N) < 2$. Hence $D(M) < 2$. $\square$

It is also useful to note that $D(M)$ and $\Lambda(M)$ behave monotonically under passage to covers.

Proposition 2.6. *Let M and M' be compact, orientable, hyperbolizable 3-manifolds, such that the interior of M' covers the interior of M. Then $D(M) \geq D(M')$ and $\Lambda(M) \leq \Lambda(M')$.*

Proof. If $N = \mathbf{H}^3/\Gamma$ is any hyperbolic 3-manifold homeomorphic to the interior of M, then it has a cover $N' = \mathbf{H}^3/\Gamma'$ which is homeomorphic to the interior of N'. Since $\Gamma' \subset \Gamma$, $L_{\Gamma'} \subset L_\Gamma$, so $d(N) \geq d(N')$. The assertion that $D(M) \geq D(M')$ then follows immediately from the definition of our invariant D.

The proof of the assertion that $\Lambda(M) \leq \Lambda(M')$ is similar. $\square$

2.3. **Outline of proof of the main theorems.** We will break the argument up into several steps. First we reduce to considering manifolds with incompressible boundary.

We say that M is obtained from two manifolds M_0 and M_1 by *adding a 1-handle* if M is obtained from M_0, M_1 and $D^2 \times [0,1]$ by identifying $D^2 \times \{i\}$ with an embedded disk in ∂M_i (for $i = 0, 1$.) M is said to be obtained from M_0 by adding a 1-handle if M is obtained from M_0 and $D^2 \times [0,1]$ by identifying $D^2 \times \{0\}$ and $D^2 \times \{1\}$ with disjoint embedded disks in ∂M_0. M is said to be obtained from $\{M_1, \ldots, M_n\}$ by adding 1-handles if it is obtained by applying the above two topological

operations finitely many times using the manifolds $\{M_1, \dots, M_n\}$ as building blocks.

Bonahon [5] and McCullough-Miller [22] showed that if M is a compact irreducible 3-manifold, then there exists a collection $\{M_1, \dots, M_n\}$ of submanifolds of M such that M is obtained from $\{M_1, \dots, M_n\}$ by adding 1-handles and each M_i has incompressible boundary. (The boundary of a 3-manifold M is incompressible if the fundamental group of any component injects in $\pi_1(M)$). The union $\cup M_i$ is called the *incompressible core* of M. If M is a handlebody its incompressible core is a ball, and otherwise we will assume that no component of the incompressible core is a ball. With this convention, the incompressible core is unique up to isotopy.

In section 3 we will use work of Patterson [27] to show that one may analyze our invariants on M simply by studying the invariants of the components of the incompressible core of M:

Theorem 2.7. *Let M be a compact, orientable, hyperbolizable 3-manifold. If $\{M_1, \dots, M_n\}$ are the components of the incompressible core of M, then*

$$\Lambda(M) = \min\{\Lambda(M_1), \dots, \Lambda(M_n)\}.$$

The idea is to "pull apart" the groups uniformizing the components of the incompressible core. For example, if M is obtained from M_0 and M_1 by adding a 1-handle, we first find $N_0 \in TT(M_0)$ and $N_1 \in TT(M_1)$ with $\lambda_0(N_i)$ near $\Lambda(M_i)$. We then construct a sequence $\{N^i\}$ of hyperbolic 3-manifolds homeomorphic to the interior of M by removing half-spaces from N_0 and N_1 which lie farther and farther from a fixed point in each, and gluing the boundary planes together. Patterson's result is used to show that $\{\lambda_0(N^i)\}$ converges to $\min\{\lambda_0(N_0), \lambda_0(N_1)\}$ and hence that $\Lambda(M) = \min\{\Lambda(M_0), \Lambda(M_1)\}$.

We now turn to 3-manifolds with incompressible boundary, for which we will need a bit more notation. If X is an annulus or torus, we say that a map $f : (X, \partial X) \to (M, \partial M)$ is *essential* if $f_* : \pi_1(X) \to \pi_1(M)$ is injective and f is not properly homotopic to a map of X into M with image in ∂M. We will say that an embedding $f : R \to M$ of an I-bundle R into M is *admissible* if $f^{-1}(\partial M)$ is the associated ∂I-bundle of R. We will say that an embedding $f : R \to M$ of a Seifert-fibred space R into M is admissible if $f^{-1}(\partial M)$ is a collection of fibres in ∂S. In either case, we will say that f is *essential* if $f_* : \pi_1(R) \to \pi_1(M)$ is injective and whenever X is a component of $\partial R - f^{-1}(\partial M)$ then $f|_X$ is an essential map of a torus or annulus into M.

A compact submanifold Σ of M is said to be a *characteristic submanifold* if Σ consists of a minimal collection of admissibly embedded

essential I-bundles and Seifert fibre spaces with the property that every essential, admissible embedding $f : R \to M$ of a Seifert fibre space or I-bundle into M is properly homotopic to an admissible map with image in Σ. Jaco-Shalen [17] and Johannson [18] showed that every compact, orientable, irreducible 3-manifold with incompressible boundary contains a characteristic submanifold and that any two characteristic submanifolds are isotopic. Hence, we often speak of *the* characteristic submanifold $\Sigma(M)$ of M. If M is hyperbolizable then every Seifert fibred component of $\Sigma(M)$ is homeomorphic to either a solid torus or a thickened torus (see Morgan [24]).

We recall that a compact irreducible 3-manifold M with incompressible boundary is called a *generalized book of I-bundles* if one may find a disjoint collection A of essential annuli in M such that each component R of the manifold obtained by cutting M along A is either a solid torus, a thickened torus, or homeomorphic to an I-bundle such that $\partial R \cap \partial M$ is the associated ∂I-bundle. One may check that a compact, orientable, irreducible 3-manifold with incompressible boundary is a generalized book of I-bundles if the closure of any component of $M - \Sigma(M)$ is homeomorphic to a solid torus or a thickened torus, and every component of $\Sigma(M)$ is a solid torus, a thickened torus, or an I-bundle (see Proposition 4.3 in [14] for the corresponding characterization of books of I-bundles.)

In section 4 we prove:

Theorem 2.8. *If M is a hyperbolizable generalized book of I-bundles, then $\Lambda(M) = 1$.*

The proof is by explicit construction. We build a hyperbolic structure for M by piecing together structures on each of the I-bundle pieces, and show that if the parameters are chosen appropriately (essentially "pulling apart" the I-bundles) the Hausdorff dimension d can be made arbitrarily close to 1. The result then follows via the connection between D and Λ.

In section 6 we prove:

Theorem 2.9. *If M is a compact, orientable, hyperbolizable 3-manifold with incompressible boundary which is not a generalized book of I-bundles, then $\Lambda(M) < 1$.*

Here is an outline of the proof in the case that M is acylindrical. Arguing by contradiction, we assume the existence of a sequence $N_i \in TT(M)$ with $\lambda_0(N_i) \to 1$. By a theorem of Thurston the deformation space of M is compact and we can extract a limit manifold N which is homeomorphic to the interior of M with $\lambda_0(N) = 1$. This contradicts

the assumption that M is not a handlebody, thickened torus or I-bundle, by Theorem 2.2.

If M is not acylindrical, we must apply the Jaco-Shalen-Johannson characteristic submanifold theory, and Thurston's relative compactness theorem. In general, the limit manifold we obtain will be homeomorphic to a submanifold of M, rather than to M itself.

Theorems 2.7, 2.8 and 2.9 combine to give a complete proof of Main Theorem I. Main Theorem II follows immediately from Main Theorem I and corollary 2.3.

In section 7 we will make further comments and conjectures about the invariants and related quantities.

3. Reduction to the incompressible case

In this section we prove Theorem 2.7, which assures us that our invariants are determined by their value on the components of the incompressible core. We do this by showing that if M is obtained by adding a 1-handle to M_0 and M_1 (or by adding a 1-handle to M_0) then $\Lambda(M) = \min\{\Lambda(M_0), \Lambda(M_1)\}$ (or $\Lambda(M) = \Lambda(M_0)$).

The topological operation of adding a 1-handle is realized geometrically by Klein combination. Throughout this section we work in the ball model of $\mathbf{H}^3$; the Euclidean boundary in this model is S^2, and $\overline{\mathbf{H}^3} = \mathbf{H}^3 \cup S^2$. If $F \subset \mathbf{H}^3$ is a (convex) *fundamental polyhedron* of Γ, then $int(F)$ denotes the interior of F, $\overline{F}$ denotes the Euclidean closure of F, and $F^c = \mathbf{H}^3 - F$. We refer the reader to sections IV.F and VI.A of [21] for a full discussion of fundamental polyhedra.

Theorem 3.1. *(Klein Combination, Theorem VII.A.13 in* [21]*) Let Γ_0 and Γ_1 be discrete subgroups of $PSL_2(\mathbf{C})$. Suppose there are (convex) fundamental polyhedra F_i for Γ_i $(i = 0, 1)$, so that $int(F_0) \cup int(F_1) = \mathbf{H}^3$. Then the group Γ generated by Γ_0 and Γ_1 is discrete and is isomorphic to $\Gamma_0 * \Gamma_1$. Moreover, $F = F_0 \cap F_1$ is a fundamental polyhedron for Γ.*

If Γ_0 and Γ_1 satisfy the hypotheses of Theorem 3.1 we will say that they are *Klein-combinable*.

The main tool in the proof of Theorem 2.7 is a result of Patterson. Let Γ_0 and Γ_1 be two Klein-combinable groups and suppose $N_i = \mathbf{H}^3/\Gamma_i$. The intuitive content of Patterson's result is that, if we "pull Γ_1 away from Γ_0" by a suitable sequence of conjugations $h_k \Gamma_1 h_k^{-1}$, then λ_0 of the quotient of the combination of Γ_0 and $h_k \Gamma_1 h_k^{-1}$ approaches $\min\{\lambda_0(N_0), \lambda_0(N_1)\}$.

The statement we give is a version of Theorem 1 in [27]. (Patterson actually proves his result for the critical exponents of the groups

involved. However, see [30], the critical exponent of Γ determines $\lambda_0(\mathbf{H}^3/\Gamma)$ and we have translated Patterson's result into a result about λ_0.) Let $|g^{'}(x)|$ denote the Euclidean norm of the derivative of g at x, where $g \in \mathrm{PSL}_2(\mathbf{C})$ and $x \in \mathbf{H}^3$. Let $d(A_1, A_2)$ denote the Euclidean distance between sets A_1 and A_2 in $\overline{\mathbf{H}^3}$.

Theorem 3.2. *(Patterson) Let Γ_0 and Γ_1 be discrete, torsion-free subgroups of $PSL_2(\mathbf{C})$ with convex fundamental polyhedra F_0 and F_1. Let $N_i = \mathbf{H}^3/\Gamma_i$. Suppose there is a sequence $\{h_k\}_{k \in \mathbf{Z}_+}$ in $PSL_2(\mathbf{C})$ so that $int(F_0) \cup h_k(int(F_1)) = \mathbf{H}^3$ for all k and*

$$\frac{\sup_{w \in F_1^c} |h_k^{'}(w)|}{d(F_0^c, h_k F_1^c)} \to 0$$

as $k \to \infty$. Let Γ^k be the discrete group generated by Γ_0 and $h_k \Gamma_1 h_k^{-1}$ and let $N^k = \mathbf{H}^3/\Gamma^k$. Then

$$\lim_{k \to \infty} \lambda_0(N^k) = \min\{\lambda_0(N_0), \lambda_0(N_1)\}.$$

We begin by studying the case where M is obtained from M_0 and M_1 by adding a 1-handle. Combining Theorems 3.1 and 3.2, we obtain:

Proposition 3.3. *Let M_0 and M_1 be two compact, orientable, hyperbolizable 3-manifolds. If M is hyperbolizable and is obtained from M_0 and M_1 by adding a 1-handle then $\Lambda(M) = \min\{\Lambda(M_0), \Lambda(M_1)\}$.*

Proof. We first note that Proposition 2.6 implies that

$$\Lambda(M) \le \min\{\Lambda(M_0), \Lambda(M_1)\},$$

since the interior of M is covered by both the interior of M_0 and the interior of M_1.

Without loss of generality assume $\Lambda(M_0) = \min\{\Lambda(M_0), \Lambda(M_1)\}$. Then Proposition 2.5 ensures that $\Lambda(M_0) > 0$. Fix a positive $\epsilon < \Lambda(M_0)$, and choose $N_i = \mathbf{H}^3/\Gamma_i \in TT(M_i)$ so that $\lambda_0(N_1) \ge \lambda_0(N_0) > \Lambda(M_0) - \epsilon > 0$. Canary [11] showed that if a complete hyperbolic 3-manifold is topologically tame but not geometrically finite, then $\lambda_0 = 0$. Therefore, N_0 and N_1 are both geometrically finite.

It follows that there exist homeomorphisms $\psi_i : \hat{N}_i \to M_i - P_i$, where $\hat{N}_i = (\mathbf{H}^3 \cup \Omega_{\Gamma_i})/\Gamma_i$ and P_i is a collection of disjoint incompressible annuli and tori in ∂M_i. Because M is hyperbolizable, the attaching disks of the 1-handle are not contained in toroidal boundary components of M_i. Hence, we may choose disks D_i in $M_i - P_i$, such that M is obtained from M_0 and M_1 by attaching a 1-handle to the disks D_0 and D_1. We can also assume that the pre-images $\psi_i^{-1}(D_i)$ are "round" disks (i.e. they are the quotients of round disks in Ω_{Γ_i}.)

We choose lifts $\widetilde{D}_i$ of $\psi^{-1}(D_i)$ to Ω_{Γ_i}. We may assume, by conjugating Γ_1, that the round disks $\widetilde{D}_0$ and $\widetilde{D}_1$ intersect only along their common boundary circle J. One can find (convex) fundamental polyhedra F_i for Γ_i such that $\widetilde{D}_i$ are contained in the interiors of the intersection of the Euclidean closures of F_i with S^2. Therefore $H_i \subset int(F_i)$, where H_i denotes the closed half-space for which $\overline{H_i} \cap S^2 = \widetilde{D}_i$. Thus, Γ_0 and Γ_1 are Klein-combinable. Since $F_0 \cap F_1$ is a fundamental polyhedron for the group Γ generated by Γ_0 and Γ_1, we see that $N = \mathbf{H}^3/\Gamma \in TT(M)$.

Fix points z_i in the interior of $\widetilde{D}_i$. Let γ be a hyperbolic Möbius transformation with z_0 as its attracting fixed point and z_1 as its repelling fixed point. We may further choose γ so that $\gamma(\widetilde{D}_0)$ is contained in the interior of $\widetilde{D}_0$. We will apply Theorem 3.2 to the sequence $\{h_k = \gamma^k\}$.

Note that $\gamma^k(\widetilde{D}_0) \subset \gamma(\widetilde{D}_0) \subset int\widetilde{D}_0$ for all $k \geq 1$ which implies that $\gamma^k(H_0) \subset \gamma(H_0) \subset H_0$. Since $\gamma^k(F_1)$ is a fundamental polyhedron for $\gamma^k\Gamma_1\gamma^{-k}$ and $\gamma^k(F_1)^c$ is contained in $\gamma^k(H_0) \subset int(H_0) \subset F_0$, we see that Γ_0 and $\gamma^k\Gamma_1\gamma^{-k}$ are Klein-combinable and that $F_0 \cap \gamma^k(F_1)$ is a fundamental polyhedron for the group Γ^k generated by Γ_0 and $\gamma^k\Gamma_1\gamma^{-k}$. In particular, one may readily observe that $N^k = \mathbf{H}^3/\Gamma_k \in TT(M)$ for all k.

Since $F_0^c \subset H_1$ and $\gamma^k(F_1)^c \subset \gamma^k(H_0) \subset \gamma(H_0) \subset int(H_0)$, we see that $d(F_0^c, \gamma^k(F_1^c)) \geq \delta$ for all $k \geq 1$, where $\delta = d(H_1, \gamma(H_0)) > 0$.

One may easily check that $\{(\gamma^k)'\}$ converges to 0 uniformly on all compact subsets of $\overline{\mathbf{H}^3} - \{z_1\}$. Since $F_1^c \subset H_0$ and $\overline{H_0}$ is a compact subset of $\overline{\mathbf{H}^3} - \{z_1\}$, then $\sup_{w \in F_1^c} |(\gamma^k)'(w)|$ converges to 0.

We may combine the observations above to establish that

$$\frac{\sup_{w \in F_1^c} |(\gamma^k)'(w)|}{d(F_0^c, h_k F_1^c)} \to 0.$$

Theorem 3.2 then allows us to conclude that

$$\lim_{k \to \infty} \lambda_0(N^k) = \min\{\lambda_0(N_0), \lambda_0(N_1)\} \geq \Lambda(M_0) - \epsilon.$$

Therefore, $\Lambda(M) \geq \min\{\Lambda(M_0), \Lambda(M_1)\} - \epsilon$. Since ϵ can be arbitrarily small, this completes the proof of Proposition 3.3. ☐

We will also need the following direct analogue of Theorem 1 in [27], which can be deduced from Patterson's arguments.

Theorem 3.4. *(Patterson) Let Γ_0 be a torsion-free discrete subgroup of $PSL_2(\mathbf{C})$ with convex fundamental polyhedron F_0, and let $\{h_k\}$ be an infinite sequence in $PSL_2(\mathbf{C})$. Suppose that F_k is a convex fundamental polyhedron for $\langle h_k \rangle$ such that $int(F_0) \cup int(F_k) = \mathbf{H}^3$ for all k, and there*

exists a $\delta > 0$ so that $d(F_0^c, F_k^c) \geq \delta$. Also, assume there exists a w lying in $F_0 \cap F_k$ for all index k, such that for any fixed $s > 0$,

$$\sum_{j \neq 0} |(h_k^j)'(w)|^s \to 0$$

as $k \to \infty$. Denote by Γ^k the discrete group generated by Γ_0 and $\langle h_k \rangle$, and let $N_0 = \mathbf{H}^3/\Gamma_0$ and $N^k = \mathbf{H}^3/\Gamma^k$. Then

$$\lim_{k \to \infty} \lambda_0(N^k) = \lambda_0(N_0).$$

Theorem 3.4 allows us to handle the case where M is obtained from M_0 by adding a 1-handle.

Proposition 3.5. *Let M_0 be a compact, irreducible, hyperbolizable* 3-*manifold with non-empty boundary. If M is a compact hyperbolizable 3-manifold obtained by adding a 1-handle to M_0, then $\Lambda(M) = \Lambda(M_0)$.*

Proof. As in Proposition 3.3, we observe that Proposition 2.6 implies that $\Lambda(M) \leq \Lambda(M_0)$ and Proposition 2.5 guarantees that $\Lambda(M_0) > 0$.

Fix a positive $\epsilon < \Lambda(M_0)$, and choose $N_0 \in TT(M_0)$ such that $\lambda_0(N_0) \geq \Lambda(M_0) - \epsilon$. As before, by [11], N_0 is necessarily geometrically finite. Thus, there exists a collection P of disjoint incompressible annuli and tori in ∂M_0 and a homeomorphism $\psi : \hat{N}_0 \to M_0 - P$, where $\hat{N}_0 = (\mathbf{H}^3 \cup \Omega_{\Gamma_0})/\Gamma_0$. Since M is hyperbolizable, the attaching disks of the 1-handle are not contained in toroidal boundary components of M. Hence, we may choose disks D_0 and D_1 in $\partial M_0 - P$ such that M is formed by attaching a 1-handle to the disks D_0 and D_1. Since the interior of $\bar{F} \cap \Omega_{\Gamma_0}$ is a fundamental domain for the action of Γ_0 on Ω_{Γ_0} (Proposition VI.A.3 in [21]), we may further choose D_0 and D_1 so that there are lifts $\widetilde{D}_0$ and $\widetilde{D}_1$ of $\psi^{-1}(D_0)$ and $\psi^{-1}(D_1)$ which are round disks in the interior of $\bar{F} \cap \Omega_{\Gamma_0}$.

Find a loxodromic element γ that takes the exterior of $\widetilde{D}_0$ to the interior of $\widetilde{D}_1$. Let H_i be the closed half-spaces whose Euclidean closures intersect S^2 in $\widetilde{D}_i$. Then the region $F_k = \mathbf{H}^3 - (H_0 \cup \gamma^{k-1}(H_1))$ is a fundamental polyhedron for $\langle \gamma^k \rangle$. Since $int(F_k) \cup int(F_0) = \mathbf{H}^3$, Γ_0 and $\langle \gamma^k \rangle$ are Klein-combinable and $F_k \cap F_0$ is a convex fundamental polyhedra for the group Γ^k generated by Γ and $\langle \gamma^k \rangle$. It is now easy to check that $N_k = \mathbf{H}^3/\Gamma^k \in TT(M)$.

We will apply Theorem 3.4 with $\{h_k = \gamma^k\}$. Let $\delta = d(F_0^c, H_0 \cup H_1) > 0$ (recall $H_i \subset F_0$). Because the H_i are disjoint and $F_k^c \subset H_0 \cup H_1$, then

$$d(F_0^c, F_k^c) \geq \delta$$

for all k.

Fix $w \in F_0 \cap F_k$ for all $k > 0$ and fix $s > 0$. It is well-known that $\sum_{j\neq 0} |(\gamma^j)'(w)|^s$ is finite. It follows immediately that $\sum_{j\neq 0} |(h_k^j)'(w)|^s = \sum_{j\neq 0} |(\gamma^{jk})'(w)|^s$ converges to 0 as k converges to ∞.

Thus, Theorem 3.4 implies that

$$\lim_{k\to\infty} \lambda_0(N^k) = \lambda_0(N_0).$$

Therefore, $\Lambda(M) > \Lambda(M_0) - \epsilon$. Since $\epsilon > 0$ was chosen arbitrarily, we have completed the proof of proposition 3.5. □

Notice that one need only apply Propositions 3.3 and 3.5 finitely many times in order to prove Theorem 2.7.

4. Generalized books of I-bundles

In this section we will prove Theorem 2.8, which says that $\Lambda(M) = 1$ for any hyperbolizable generalized book of I-bundles M. The key step in the proof is:

Theorem 4.1. *If M is a hyperbolizable generalized book of I-bundles then for any $\alpha > 1$, there exists a hyperbolic manifold N homeomorphic to int(M) with $d(N) < \alpha$.*

Proof of Theorem 2.8. Let M be a hyperbolizable generalized book of I-bundles which is not a thickened torus. Theorem 4.1 implies that $D(M) \leq 1$. On the other hand, since M is not a handlebody or a thickened torus, Corollary 2.4 guarantees that $D(M) \geq 1$. Hence, $D(M) = 1$ and we conclude that $\Lambda(M) = 1$ by applying Corollary 2.3. If M is a thickened torus we have already observed that $\Lambda(M) = 1$. □

The remainder of the section is taken up with the proof of Theorem 4.1.

Proof of Theorem 4.1. The characteristic submanifold of M is a union of solid tori, thickened tori, and I-bundles whose bases have negative Euler characteristic. For each I-bundle the subbundle over the boundary of the base surface is a union of annuli, which are glued to the boundary of a solid torus or thickened torus (for a thickened torus, note that only one of its boundaries participates in the gluing). The union of the bases of the I-bundles, with boundaries glued together inside each solid torus, and torus boundaries of the thickened tori, comprise a "spine" for M, that is a 2-complex of which M is a regular neighborhood.

We shall put a hyperbolic structure on the interior of M, for which each I-bundle base determines a Fuchsian or extended Fuchsian group, and each thickened torus corresponds to a rank-2 parabolic group. (We

recall that a discrete subgroup of $\mathrm{PSL}_2(\mathbf{C})$ is *Fuchsian* if it preserves a half-space in $\mathbf{H}^3$ and *extended Fuchsian* if it has a Fuchsian subgroup of index 2. In either case there is a totally geodesic hyperplane preserved by the group.) By changing the parameters of this construction we will obtain Hausdorff dimension arbitrarily close to 1.

For each solid torus, the cores of annuli glued to it describe some number m of parallel (p, q) curves, for some relatively prime p, q (where $(1, 0)$ denotes a meridian and p is well-defined mod q). Consider the following hyperbolic structure on this solid torus: Begin with a geodesic L in $\mathbf{H}^3$ which is the boundary of mq half-planes equally spaced around it (more generally the angles between them can vary, but we will avoid this for ease of exposition). Let γ be a loxodromic with axis L, translation distance ℓ/q, for some (small) $\ell > 0$, and rotation angle $2\pi p/q$. The quotient of a neighborhood of L by γ is a solid torus, which the quotients of the half-planes meet in a collection of annuli with boundaries glued together at the core. The intersection of these annuli with the torus boundary give the m desired (p, q) curves.

For each thickened torus we choose (large) $d > 0$ and consider a horoball in $\mathbf{H}^3$ with a rank 2 parabolic group acting so that a fundamental domain on the boundary is a rectangle with one sidelength μ_0 and one sidelength $md > 0$. In the horoball we consider m planes orthogonal to the boundary, parallel to the μ_0 side, and equally spaced (by distance d along the boundary). In the quotient these give m parallel cusps with boundary length μ_0. Here μ_0 denotes a fixed number less than the Margulis constant for $\mathbf{H}^2$.

Choose a list of parameters $\{\ell_i\}$ for the solid tori and $\{d_i\}$ for the thickened tori. For each base surface S, let S' denote S minus the boundary components that attach to thickened tori, and choose a finite-area hyperbolic structure on S' so that a neighborhood of each missing boundary component is a cusp, and each remaining boundary component that glues to a solid torus with parameter ℓ_i is a geodesic of length ℓ_i. For each base surface we can find a Fuchsian or extended Fuchsian group such that the convex core of its quotient realizes the given hyperbolic structure. (The convex core of a hyperbolic manifold is the quotient of the convex hull of the limit set by the associated group action.) Note that the boundary components correspond to pure translations. We then truncate each cuspidal end so that the boundaries corresponding to thickened tori are horocycles of length μ_0. We may obtain an incomplete structure on each I-bundle by considering the embedding of our truncated region in the full quotient of the associated Fuchsian or extended Fuchsian group.

For each solid torus we can then identify neighborhoods of the corresponding boundaries of I-bundle bases to the annuli arranged around its core, and for each thickened torus we can glue the horocycles to the boundaries of the cusps embedded in the horoball. This extends consistently to the thickenings of the I-bundle bases so that we obtain an (incomplete) hyperbolic structure on the interior of M, in which each I-bundle base is totally geodesic. With proper choice of the parameters, we will show that this gives rise to a complete structure.

The developing map for this structure maps the universal cover $\widetilde{M}$ to $\mathbf{H}^3$ by a locally isometric immersion (see e.g. Benedetti-Petronio [3] §B.1). Let Γ denote the holonomy group. Each component of the lift $\widetilde{S}$ of a base surface S maps to a totally geodesic subset of $\mathbf{H}^3$. These subsets, which we will call *flats*, are arranged in a "tree", in this sense: For a given flat F, at each lift of a geodesic boundary of its base surface there is a collection of $mq-1$ other flats, equally spaced (where m is the number of annuli glued to the corresponding torus, and (p,q) describes the slopes of these annuli, as above). At each parabolic fixed point corresponding to one of the boundaries glued to a thickened torus, there is an bi-infinite sequence of other flats, arranged with equal spacing around a horoball based at this point. The corresponding graph of adjacencies is a tree (of infinite valence).

4.1. **Discrete holonomy.** Let $\ell_0 = \max \ell_i$ and $d_0 = \min d_i$. Let $\theta_0 = \min 2\pi/q_i m_i$ where $\{m_i\}$ and $\{(p_i,q_i)\}$ describe the gluings for the solid tori. We will show that, if ℓ_0 is sufficiently small and d_0 sufficiently large, the holonomy group Γ is discrete, and the quotient manifold is homeomorphic to M.

We first make the following geometric observation, which is a standard type of fact for broken geodesics in $\mathbf{H}^n$.

Lemma 4.2. *Given $\theta \in (0,\pi]$ there exists $K \geq 0$ such that the following holds. Let γ be a broken geodesic in $\mathbf{H}^3$ composed of a chain of n segments $\gamma_1,\dots,\gamma_n$ of lengths $k_i > K$ that meet at angles $\theta_i \geq \theta$. Let P_i denote the orthogonal bisecting plane to γ_i. Then the P_i are all disjoint, and each P_j separates P_i and γ_i from P_k and γ_k whenever $i<j<k$. Furthermore* $\mathrm{dist}(P_i,P_{i+1}) \geq \frac{1}{2}(k_i+k_{i+1}) - K$.

Proof. Choose K by the formula

$$\cosh^2 K/2 = \frac{2}{1-\cos\theta}.$$

A little hyperbolic trigonometry shows that if two segments meet at their endpoints at angle θ then the planes orthogonal to the segments

at a distance $K/2$ from the intersection point meet at a single point at infinity, and if the angle is greater than θ the planes are disjoint.

Now consider for each segment of γ the family of planes orthogonal to it, excluding the ones closer than $K/2$ to either endpoint (a nonempty family since $k_i > K$). The planes meeting any segment thus separate the planes meeting the previous segment from those meeting the next segment, and the distance between the first and last plane for segment i is $k_i - K$. The statement for the bisecting planes follows from this. $\square$

Recall that the *μ-thin part* of a flat F denotes the points where some element of the stabilizer of F acts with translation μ or less. If μ is smaller than the Margulis constant, this set consists of a union of disjoint pieces, each of which is either a horodisk around a parabolic fixed point or a neighborhood of an axis of a translation. The μ-thick part is the complement of the μ-thin part.

Given any two points x, y in two flats F, F', let $F = F_1, \dots, F_n = F'$ denote the sequence of flats in the tree connecting them. Each F_i and F_{i+1} share either a geodesic boundary or a parabolic fixed point at infinity, called $F_i \cap F_{i+1}$ in either case. There is a chain of geodesics $\{\alpha_i\}$ connecting x to y such that $\alpha_i \subset F_i$, and α_i meets α_{i+1} at $F_i \cap F_{i+1}$ (possibly at infinity). The chain is uniquely determined by the condition that, whenever $F_i \cap F_{i+1}$ is a geodesic boundary, α_i meets it orthogonally. Whenever α_i, α_{i+1} meet in a parabolic point, adjust them as follows: Truncate each at the point where it enters the μ_1-horoball of the corresponding parabolic group ($\mu_1 < \mu_0$ will be determined shortly), and join the new endpoints with a geodesic, which we note makes an angle greater than $\pi/2$ with α_i and α_{i+1}. When $i = n-1$, α_n may be entirely contained in the μ_1-horoball, and in that case we remove α_n entirely and join y directly to the truncated α_{n-1}. Call the resulting chain of geodesics $\gamma_{x,y}$.

Suppose that x is in the μ_0-thick part of F. We claim that, given any k, if ℓ_0 and μ_1 are sufficiently small and d_0 is sufficiently large, each segment of $\gamma_{x,y}$, except possibly the last, has length at least k. By the collar lemma for hyperbolic surfaces, if ℓ_0 and μ_1 are sufficiently small then the μ_0-thick part of each quotient surface is separated from its boundary by at least $c \log \mu_0/\ell_0$, and from the μ_1-thin parts of the cusps by $c \log \mu_0/\mu_1$, for a fixed constant c. This bounds from below the length of each segment in $\gamma_{x,y}$, except possibly the last segment containing y, and the additional segments added in horoballs. Each segment of the latter type has length at least $c \log d_0\mu_1/\mu_0$ (for a constant c) since the horospherical distance between flats on the boundary of the μ_0-horoball is a multiple of d_0 by construction. Thus, choosing

d_0 large enough (after the choice of μ_1 is made) this gives a high lower bound for the horoball segments, and establishes our claim.

Any two segments meet at angle at least $\theta = \min\{\theta_0, \pi/2\}$, so let $K = K(\theta)$ be the constant given in Lemma 4.2 and suppose $k \geq 2K$ and ℓ_0, d_0 and μ_1 are determined as above. Lemma 4.2 then provides a sequence of planes with definite spacing that separate x from y.

In particular we can deduce that any two flats which are non-adjacent in the tree are disjoint, and more generally, fixing x in the μ_0-thick part of F, for any path $F = F_1, \ldots, F_n$ of successively adjacent flats in the tree, that

$$\mathrm{dist}(x, F_n) \geq (n-2)(k-K) + k - K/2.$$

In particular the entire tree of flats is (properly) embedded in $\mathbf{H}^3$, and therefore Γ is discrete.

It remains to show that $N = \mathbf{H}^3/\Gamma$ is homeomorphic to M. Bonahon's theorem [6] guarantees that N is topologically tame (we could also deduce this directly by showing that Γ is geometrically finite). Since a neighborhood of the tree of flats embeds, it must be the homeomorphic developing image of a neighborhood of the lift to the universal cover of the spine of M. It follows that M embeds in N, by a map which is a homotopy equivalence. By a theorem of McCullough-Miller-Swarup [23], this implies that N is homeomorphic to the interior of M.

Remark: Another approach to this construction is by means of Klein-Maskit combinations (see Maskit [20].)

4.2. **Hausdorff dimension.** We next show that, with further restrictions on ℓ_0 and d_0, we can obtain upper bounds on Hausdorff dimension. This will be done directly, by exhibiting an appropriate family of coverings.

Choose one flat F_0 as the root of the tree. Choose a point x_0 in the μ_0-thick part of F_0. Normalize the picture in the upper half-space model so that the plane H_0 containing F_0 is a hemisphere meeting the complex plane in the unit circle C_0, and so that x_0 is the point $(0,0,1)$. Each child F' of F_0 in the tree structure is of one of two types: Type (1): if F' meets F_0 along a geodesic L, F' is contained in a half-plane H' which meets the complex plane in an arc C' of a circle. There are a finite number (at most $2\pi/\theta_0$) of other flats adjoined at L. Type (2): If F' meets F_0 along a parabolic fixed point, it is part of a sequence of flats $\{F_n\}_{n\in\mathbf{Z}}$ meeting at that point, where F_n are all children of F_0 for $n \neq 0$. These meet the plane in a family of concentric circles $\{C_n\}$ tangent at the same point. These divide into C_0 itself, and the circles

outside and inside. Call the set of outer ones (and of inner ones) an "earring".

The same description holds for the children of any flat. We thus get a family of circular arcs $\{C\}$ arranged in a tree structure with root C_0. (Note that for type (1) flats the arc is just the portion of a circle where a half-plane meets $\mathbf{C}$, whereas for type (2) it is a whole circle.) For any C let $s(C)$ denote the set of its children, and similarly $s^2(C) = s(s(C))$, etc. Let $r(C)$ denote the diameter of C, which of course is uniformly comparable to the length of C.

The limit set L_Γ of Γ is contained in the closure $\hat{L}_\Gamma$ of the union of these arcs C.

Fix any positive $\rho < 1/2$. We claim that we can choose the parameter d_0 sufficiently large and ℓ_0 sufficiently small, so that the following holds (where c_0 is a fixed constant):

- For any C and $D \in s(C)$,
$$r(D) \leq \rho r(C). \tag{4.1}$$
- If $D_1, D_2, \ldots$ are the nested circles in an earring, with D_1 the outermost, then
$$r(D_n) \leq c_0 \frac{r(D_1)}{n}. \tag{4.2}$$

Proof of (4.1). Let $\mu_2 < \mu_0$ be a constant to be chosen later. Let H and H' be the hemispheres containing C and D, respectively. They meet either in a geodesic g or at a parabolic fixed point. Let $J(H, H')$ denote the component of the μ_2-thin part associated to the intersection in either case. Let x be the top point of H (in the upper half-space). Suppose that x is outside $J(H, H')$. Then the geodesic chain $\gamma_{x,y}$ for any $y \in H'$, constructed as in §4.1 but with μ_2 taking the place of μ_0, has initial segment γ_1 of length at least k, where k can be made arbitrarily large by making ℓ_0/μ_2 and μ_1/μ_2 small, and d_0 large. Given these choices, we conclude via Lemma 4.2 that all of H' is separated from x by the bisecting hemisphere of γ_1, which is distance at least $k/2$ from x, and is thus of Euclidean diameter at most $ce^{-k/2} \operatorname{diam}(H)$ for a fixed c. This gives the desired bound on $r(D)$, if k is chosen so that $ce^{-k/2} \leq \rho$.

It remains to show that, with appropriate choice of μ_2, the top x of each H is outside $J(H, H')$ for any child H' of H. For the root of the tree this holds by our normalization. We argue by induction. Suppose that H_1, H_2, H_3 are hemispheres such that H_i is the parent of H_{i+1} and x_i are the tops of H_i. If the inductive hypothesis holds for H_1 then, in particular, $\operatorname{diam}(H_2) \leq \operatorname{diam}(H_1)$ by the above paragraph.

It follows that $\mathrm{dist}(x_2, H_1)$ is bounded from above by a fixed number a. However, if x_2 were contained in $J(H_2, H_3)$ then there would be a separation between x_2 and $J(H_1, H_2)$ of at least $c\log(\mu_0/\mu_2)$, since $J(H_1, H_2)$ and $J(H_2, H_3)$ meet H_2 in two distinct, and hence disjoint, components of the μ_0 thin part. Assuming that μ_2 is sufficiently short this distance is long enough that we can construct a geodesic chain $\gamma_{x_2,y}$ from x_2 to any $y \in H_1$ with long initial segment, and apply Lemma 4.2 to conclude $\mathrm{dist}(x_2, H_1) > a$, a contradiction. Thus there is an a-priori choice of μ_2 which guarantees that x_2 will be outside $J(H_2, H_3)$, and we are done by induction. □

Proof of (4.2). We observe that since, by (4.1), D_1 is at most half the size of the parent C, we may re-normalize, by a Möbius transformation whose derivative is within a universally bounded ratio of a constant on all of the D_i, so that C becomes a straight line meeting the D_i at the origin. The D_i, and C, are then taken by the map $z \mapsto 1/z$ to a sequence of equally spaced parallel lines. An easy computation gives (4.2), where the constant c_0 comes from the initial re-normalization. □

We now claim the following, for any $2 \geq \alpha > 1$:

$$\sum_{D\in s(C)} r(D)^\alpha \leq a_0\rho^{\alpha-1}\frac{\zeta(\alpha)}{2^{\alpha-1}-1}r(C)^\alpha \tag{4.3}$$

where a_0 is a fixed constant and $\zeta(\alpha) = \sum 1/n^\alpha$ is the usual Zeta function.

To prove this consider first the children of type (1): these are arranged in groups of bounded number which subtend a common interval on C, and any two such intervals are disjoint. The lower bound θ_0 on the angle at which any such child meets C implies that its diameter is comparable to the diameter of the interval. The sum of lengths of intervals is at most the length of C, so there is some constant a_1 such that $\sum r(D) \leq a_1 r(C)$, for D of type (1). Since also $r(D) \leq \rho r(C)$, we make the following observation: If $\sum x_i \leq ax$ and each $x_i \leq \rho x$, then $\sum x_i^\alpha \leq \sum x_i(\rho x)^{\alpha-1} \leq a\rho^{\alpha-1}x^\alpha$. Thus we can bound the contribution of type (1) children by:

$$\sum_{\text{type 1}} r(D)^\alpha \leq a_1\rho^{\alpha-1}r(C)^\alpha. \tag{4.4}$$

For the children of type (2) the sum of lengths is infinite and we must take more care. Consider first an earring $D_1, D_2, \ldots$ with D_1

outermost. By (4.2), we have

$$\sum_{n=1}^{\infty} r(D_n)^\alpha \le c_0^\alpha \zeta(\alpha) r(D_1)^\alpha. \tag{4.5}$$

At each parabolic point p on C there are two earrings (inside and outside the circle). Let D_p denote the outermost circle of the outside earring. Clearly it just remains to bound $\sum_p r(D_p)^\alpha$.

Note first that all the D_p are disjoint, by the argument of §4.1 showing that the tree of flats embeds. It follows, we claim, for any $\delta > 0$, that

$$\sum_{\frac{r(D_p)}{r(C)} \in [\delta/2,\delta]} r(D_p)^\alpha \le a_2 \delta^{\alpha-1} r(C)^\alpha. \tag{4.6}$$

Each D_p projects radially to an interval on C, and the condition that diameters lie in $[r(C)\delta/2, r(C)\delta]$, together with disjointness of the D_p, means that these intervals cover C with multiplicity at most 2. This implies $\sum r(D_p) \le a_2 r(C)$ for this subset with some constant a_2, and (4.6) follows, using the same observation as for the type (1) children.

Summing over $\delta = \rho/2^k$ for $k = 0, 1, \ldots$, we obtain a bound for the sum over all outer circles of (outside) earrings:

$$\begin{aligned} \sum_p r(D_p)^\alpha &\le a_2 r(C)^\alpha \sum_{k=0}^{\infty} \left(\frac{\rho}{2^k}\right)^{\alpha-1} \\ &\le \frac{2a_2\rho^{\alpha-1}}{2^{\alpha-1}-1} r(C)^\alpha. \end{aligned} \tag{4.7}$$

The same argument works for the outer circles of earrings contained inside C, doubling our bound. Combining with the bound (4.5) for the sum over each earring, we obtain the inequality (4.3) for the sum over type (2) children. Now combining with (4.4) we get (4.3) over all the children of C (note that in (4.4) the factor $\zeta(\alpha)/(2^{\alpha-1}-1)$ does not appear, but this does not matter since it has a positive lower bound when $\alpha \in (1, 2]$).

We shall now define, for any $\epsilon_0 > 0$, a covering of the closure $\hat{L}_\Gamma$ of the union of arcs C, by balls of radius less than or equal to ϵ_0.

For each arc C we shall inductively assign a number $\epsilon(C)$ with which to cover C. Let $\epsilon(C_0) = \epsilon_0$. For a child D of C let

$$\epsilon(D) = \frac{r(D)}{\rho r(C)} \epsilon(C).$$

Note that $\epsilon(D) \leq \epsilon(C)$ by (4.1). Furthermore we observe by induction that if $C \in s^j(C_0)$, the j-th level of the tree, then

$$\epsilon(C) = \frac{r(C)}{\rho^j r(C_0)}\epsilon_0 \leq \epsilon_0.$$

Recall that $\rho < 1/2$. If D is a child of C and $T(D)$ is the union of all arcs in the subtree whose root is D, we immediately have by (4.1) that $T(D)$ is contained in a ball of radius

$$\sum_{k=0}^{\infty} \rho^k r(D) < 2r(D)$$

around any point of D. Thus there is some fixed a_3 for which there exists a covering of C by $a_3 r(C)/\epsilon(C)$ balls of radius $\epsilon(C)$, which also covers the closure of any subtree descended from C with root D, provided $r(D) \leq \epsilon(C)/2$.

Now given $k > 0$ let $\epsilon_0 = \rho^k r(C_0)$ and let U_k denote the covering which is the union of these coverings for all C in levels 0 through k. We claim that U_k is in fact a covering of all of $\hat{L}_\Gamma$. For, if $D \in s(C)$ and C is at level k, $r(D) \leq \rho r(C)$, and $\epsilon(C) = (r(C)/\rho^k r(C_0))\epsilon_0 = r(C)$. Thus the covering for C covers the closure of $T(D)$, as above.

We shall now compute the α-dimensional mass of this covering. Let M_k denote the sum $\sum r_i^\alpha$ over the balls of U_k, where r_i denotes the radius of the i-th ball and let $M_k(C)$ for any arc C denote the sum over just the subset of balls covering C.

For each C we have

$$M_k(C) = a_3(r(C)/\epsilon(C))\epsilon(C)^\alpha = a_3 r(C)\epsilon(C)^{\alpha-1}.$$

If $D \in s(C)$, we get

$$\begin{aligned} M_k(D) &= a_3 r(D)\left(\frac{r(D)\epsilon(C)}{\rho r(C)}\right)^{\alpha-1} \\ &= a_3\left(\frac{\epsilon(C)}{\rho r(C)}\right)^{\alpha-1} r(D)^\alpha. \end{aligned}$$

Summing over all $D \in s(C)$ and using (4.3),

$$\begin{aligned} \sum_{D\in s(C)} M_k(D) &\leq a_3\left(\frac{\epsilon(C)}{\rho r(C)}\right)^{\alpha-1} a_0\rho^{\alpha-1}\frac{\zeta(\alpha)}{2^{\alpha-1}-1}r(C)^\alpha \\ &= a_0\frac{\zeta(\alpha)}{2^{\alpha-1}-1}a_3 r(C)\epsilon(C)^{\alpha-1} \\ &= A(\alpha)M_k(C). \end{aligned} \tag{4.8}$$

Where we abbreviate $A(\alpha) = a_0 a_3 \zeta(\alpha)/(2^{\alpha-1} - 1)$. Applying this inductively, we get

$$\sum_{D \in s^j(C_0)} M_k(D) \leq A(\alpha)^j M_k(C_0) \tag{4.9}$$

For any $j \leq k$. Let us assume $A(\alpha) \geq 2$ (since we may always enlarge a_0). Then $\sum_{j=0}^{k} A(\alpha)^j \leq A(\alpha)^{k+1}$, and summing up (4.9) over levels 0 through k for the covering U_k, we get

$$M_k \leq A(\alpha)^{k+1} M_k(C_0). \tag{4.10}$$

By the choice of $\epsilon_0 = r(C_0)\rho^k$, we have

$$M_k(C_0) = a_3 r(C_0)^\alpha (\rho^{\alpha-1})^k.$$

Thus, given any $\alpha > 1$, we may choose ρ small enough that $\rho^{\alpha-1} < 1/A(\alpha)$, and then $\lim_{k\to\infty} M_k = 0$, and $\hat{L}_\Gamma$ (hence L_Γ) has zero Hausdorff measure in dimension α. Thus $d(\mathbf{H}^3/\Gamma) \leq \alpha$, which concludes the proof of Theorem 4.1. □

5. More preliminaries

In this section we recall more of the background which will be needed to handle manifolds with incompressible boundary which are not books of I-bundles. We first recall Bonahon's theorem about topological tameness and some basic facts about geometric convergence. In section 5.3 we recall Thurston's relative compactness theorem and prove an unmarked version of it which will be the key technical tool in the proof of theorem 2.9.

5.1. **Bonahon's theorem.** Bonahon [6] proved that a hyperbolic 3-manifold $\mathbf{H}^3/\Gamma$ is topologically tame if Γ satisfies Bonahon's condition (B), which is the following: whenever $\Gamma = A * B$ is a non-trivial free decomposition of Γ, there exists a parabolic element of Γ which is not conjugate to an element of either A or B.

Bonahon provided the following topological interpretation of his condition (B) (see Proposition 1.2 in [6].)

Lemma 5.1. *Let Γ be a torsion-free Kleinian group. Suppose that M is a compact 3-manifold, P is a collection of homotopically non-trivial annuli in ∂M, no two of which are homotopic in M and every component of $\partial M - P$ is incompressible in M. Then, if there exists an isomorphism $\phi : \pi_1(M) \to \Gamma$ such that $\phi(g)$ is parabolic if g is conjugate to an element of $\pi_1(P)$, then Γ satisfies Bonahon's condition (B).*

5.2. **Geometric convergence.** We say that a sequence of Kleinian groups $\{\Gamma_j\}$ converges *geometrically* to a Kleinian group Γ if for every $\gamma \in \Gamma$, there exists a sequence $\{\gamma_j \in \Gamma_j\}$ converging to γ and if any accumulation point of any sequence $\{\gamma_j \in \Gamma_j\}$ is contained in Γ. We will need the following observation, whose proof we sketch. For a complete argument, from a slightly different point of view, see Taylor [32].

Lemma 5.2. *Suppose that Γ_i is a sequence of torsion-free Kleinian groups converging geometrically to a torsion-free Kleinian group Γ. Let $N_i = \mathbf{H}^3/\Gamma_i$ and $N = \mathbf{H}^3/\Gamma$. Then*

$$\limsup \lambda_0(N_i) \leq \lambda_0(N).$$

Proof. Let $\{N_j\}$ be a subsequence of $\{N_i\}$ such that $\{\lambda_0(N_j)\}$ converges to $L = \limsup \lambda_0(N_i)$. Let f_j be a positive C^∞-function such that $\Delta f_j + \lambda_0(N_j) f_j = 0$ and let $\widetilde{f_j}$ be the lift of f_j to a map $\widetilde{f_j} : \mathbf{H}^3 \to \mathbf{R}$. We may scale $\widetilde{f_j}$ so that $\widetilde{f_j}(\vec{0}) = 1$ where $\vec{0}$ denotes the origin of $\mathbf{H}^3$. Yau's Harnack inequality [38] and basic elliptic theory guarantee that there is a subsequence of $\{\widetilde{f_j}\}$ which converges to a positive C^∞-function $\widetilde{f}$ such that $\Delta\widetilde{f} + L\widetilde{f} = 0$. Since $\widetilde{f_j}$ was Γ_j-invariant and Γ_j converges geometrically to Γ, $\widetilde{f}$ is Γ-invariant and hence descends to a C^∞-function on N such that $\Delta f + Lf = 0$. It follows that $\lambda_0(N) \geq L$. □

5.3. **Deformation theory of Kleinian groups.** In the proof of Theorem 2.9 we will consider sequences of Kleinian groups in an algebraic deformation space. Let us therefore introduce the following terminology. If G is a group, let $\mathcal{D}(G)$ denote the space of discrete, faithful representations of G into $\mathrm{PSL}_2(\mathbf{C})$. Let $AH(G) = \mathcal{D}(G)/\mathrm{PSL}_2(\mathbf{C})$ where $\mathrm{PSL}_2(\mathbf{C})$ acts by conjugation of the image. If c is a loop in M represented by the element $g \in \pi_1(M)$ and $\rho \in AH(\pi_1(M))$, then $l_\rho(c)$ is the translation length of $\rho(g)$ if $\rho(g)$ is hyperbolic and 0 if $\rho(g)$ is parabolic.

We will use the following basic lemma which relates convergence in $\mathcal{D}(G)$ and geometric convergence (see [19]).

Lemma 5.3. *Suppose that G is a torsion-free, non-abelian group and $\{\rho_j\}$ is a sequence in $\mathcal{D}(G)$ which converges to $\rho \in \mathcal{D}(G)$. Then there is a subsequence of $\{\rho_j(G)\}$ which converges geometrically to a torsion-free Kleinian group Γ which contains $\rho(G)$.*

In order to state Thurston's relative compactness theorem we need to introduce the *window* W of a compact hyperbolizable 3-manifold M with incompressible boundary. The window W consists of the I-bundle components of the characteristic submanifold $\Sigma(M)$ together with a

thickened neighborhood of every essential annulus in $\partial\Sigma(M) - \partial M$ which is not the boundary of an I-bundle component of $\Sigma(M)$. The window is itself an I-bundle over a surface w; w is called the *window base*.

The following is Thurston's relative compactness theorem for hyperbolic structures on M (see [36] or Morgan-Shalen [25].)

Theorem 5.4. *Let M be a compact, orientable, hyperbolizable 3-manifold with incompressible boundary and window W. If G is any subgroup of $\pi_1(M)$ which is conjugate to the fundamental group of a component of $M - W$ whose closure is not a thickened torus, then the image of the induced map from $AH(\pi_1(M))$ into $AH(G)$ has compact closure.*

We wish to extend this theorem to the following "unmarked" version of Thurston's theorem.

Theorem 5.5. *Let $\{\rho_i\}$ be a sequence in $AH(\pi_1(M))$. We may then find a subsequence $\{\rho_j\}$, a sequence of elements $\{\phi_j\}$ of $Out(\pi_1(M))$, and a collection x of disjoint, non-parallel, homotopically non-trivial simple closed curves in the window base w such that if G is any subgroup of $\pi_1(M)$ which is conjugate to the fundamental group of a component of $M - X$ whose closure is not a thickened torus (where X is the total space of the I-bundle over x), then $\{\rho_j \circ \phi_j|_G\}$ converges in $AH(G)$. Moreover, if c is a curve in x, then $\{l_{\rho_j\circ\phi_j}(c)\}$ converges to 0.*

The idea of the proof is the following. By Thurston's theorem, the restrictions of the representations to the complementary components of the window have convergent subsequences, and in particular the lengths of the window boundaries are bounded. Now for each element of the sequence we represent each component of the window base as a pleated surface with geodesic boundary, and observe that the hyperbolic structures induced on these surfaces, after appropriate remarkings and restriction to a subsequence, either converge or develop cusps. In the latter case we cut along the curves which become cusps (these are the family x), and argue that the representations restricted to the remaining components converge up to subsequence. Finally, we reglue along the window boundaries which did not converge to cusps, and again after restriction to a subsequence obtain the desired convergence.

Proof. The following proposition states the corresponding fact for hyperbolic surfaces. This fact is reasonably standard, and one may construct a proof using techniques described in Abikoff [1] and Harvey [16].

Proposition 5.6. *Let S be a compact surface with boundary and let $\{\psi_i\}$ be a sequence of discrete faithful representations of $\pi_1(S)$ into*

$Isom(\mathbf{H}^2)$. If there exists K such that $l_{\psi_i}(\partial S) \leq K$ for all i, then there exists a subsequence $\{\psi_k\}$ of $\{\psi_i\}$, a collection y of disjoint, homotopically distinct curves in S and a collection of homeomorphisms $h_k : S \to S$ which are the identity on ∂S such that if R is a component of $S - y$, then $\{\psi_k \circ (h_k)_*|_{\pi_1(R)}\}$ converges in $AH(\pi_1(R))$. Moreover, $\{l_{\psi_k \circ (h_k)_*}(y_0)\}$ converges to 0, for any component y_0 of y.

We begin by using Proposition 5.6 to choose x, a subsequence $\{\rho_k\}$ of $\{\rho_i\}$, and a sequence of homeomorphisms $\{f_k : w \to w\}$ such that if R is a component of $w - x$, then $\{\rho_k \circ (f_k|_R)_*\}$ converges in $AH(\pi_1(R))$.

Let $w_1, \ldots, w_m$ be the components of w. We choose the restriction of x to each component inductively, at the same time passing to subsequences. We inductively assume that we have chosen a subsequence $\{\rho_k\}$ and the restriction of x and f_k to $w_1 \cup \cdots w_{l-1}$ such that if R is a component of $(w_1 \cup \cdots \cup w_{l-1}) - x$, then $\{\rho_k \circ (f_k|_R)_*\}$ converges in $AH(\pi_1(R))$.

If w_l is an annulus or a Möbius strip, let c_l be a core curve of w_l. Thurston's relative compactness theorem implies that $l_{\rho_i}(c_l)$ is bounded above. We may then pass to a subsequence of $\{\rho_k\}$, again called $\{\rho_k\}$, such that $\{\rho_k|_{\pi_1(w_l)}\}$ converges in $AH(\mathbf{Z})$. We include ∂w_l in x if and only if $\{l_{\rho_k}(c_l)\}$ converges to 0. Let the restriction of f_k to w_l be the identity map.

Now suppose that w_l is not an annulus or a Möbius strip, in which case w_l has negative Euler characteristic. We may pass to a subsequence of $\{\rho_k\}$, again called $\{\rho_k\}$, such that if z is any boundary component of w_l then either $l_{\rho_k}(z) = 0$ for all k or $l_{\rho_k}(z) \neq 0$ for any k. Let w_l' be obtained from w_l by removing any component z of ∂w_l such that $l_{\rho_k}(z) = 0$ for all k. One can then find, for each k, a complete finite-area hyperbolic metric τ_k on w_l' with geodesic boundary, and a path-wise isometry $r_k : (w_l', \tau_k) \to N_k$ in the homotopy class determined by the restriction of ρ_k to $\pi_1(w_l)$, such that that $r_k(\partial w_l')$ is a collection of closed geodesics. (A map between Riemannian manifolds is a *pathwise isometry* if any rectifiable path in the domain is taken by the map to a path of equal length. Typically one takes each r_k to be a pleated surface, see [10] or [34]. In particular, note that neighborhoods of the missing boundaries of w_l' are cusps, which map into cusps in N_l.)

Then (w_l', τ_k) is isometric to the convex core of $\mathbf{H}^2/\Theta_k$ for some discrete subgroup Θ_k of $Isom(\mathbf{H}^2)$ and there is an induced discrete faithful representation $\psi_k : \pi_1(w_l) \to Isom(\mathbf{H}^2)$ with image Θ_k such that $l_{\psi_k}(c) \geq l_{\rho_k}(c)$ for any simple closed curve c in w_l and all k.

Thurston's relative compactness theorem guarantees that there exists K such that $l_{\psi_k}(\partial w_l') = l_{\rho_k}(\partial w_l) \leq K$ for all k. Then $\{\psi_k\}$ satisfies

the hypotheses of Proposition 5.6. Hence, after perhaps passing to another subsequence of $\{\rho_k\}$, again called $\{\rho_k\}$, we obtain a collection y of simple closed curves in w_l and a sequence of homeomorphisms $h_k : w_l \to w_l$ such that if R is a component of $w_l - y$, then $\{\psi_k \circ (h_k|_R)_*\}$ converges in $AH(\pi_1(R))$. We then append y to x and we also add to x any component z of ∂w_l such that $\{l_{\rho_k}(z)\} = \{l_{\psi_k}(z)\}$ converges to 0. Let the restriction of f_k to w_l agree with h_k. If R is a component of $w_l - x$, then the convergence of the Fuchsian representations $\{\psi_k \circ (h_k|_R)_*\}$ implies that any subsequence of $\{\rho_k \circ (h_k|_R)_*\}$ has a convergent subsequence in $AH(\pi_1(R))$. This is because, for an appropriate choice of basepoint in $r_k(R)$, the translation distances of a generating set of elements in $\rho_k \circ (h_k|_R)_*(\pi_1(R))$ are bounded by the translation distances in the Fuchsian representations (since r_k are pathwise isometries). Hence, we may pass to a further subsequence of $\{\rho_k\}$, again called $\{\rho_k\}$, such that if R is any component of $w_l - x$, then $\{\rho_k \circ (h_k|_R)_*\}$ converges in $AH(\pi_1(R))$.

Let $\{\rho_k\}$ be the subsequence and $\{f_k : w \to w\}$ be the sequence of homeomorphisms obtained after applying the above process m times. Each f_k induces a homeomorphism $F_k : W \to W$ preserving $\partial M - \partial W$, which is homotopic to the identity on each component of $\partial W - \partial M$. Hence, F_k extends to a homotopy equivalence $\hat{F}_k$ of M which is equal to the identity on the complement of a regular neighborhood of W.

Let $\phi'_k = (\hat{F}_k)_*$ and let Z be the I-bundle over $\partial w - x$. Thurston's relative compactness theorem and our construction of $Z \cup X$ imply that there exists a subsequence of $\{\rho_k\}$, again denoted $\{\rho_k\}$, such that if M' is any component of $M - (Z \cup X)$ whose closure is not a thickened torus, then $\{\rho_k \circ \phi'_k|_{\pi_1(M')}\}$ converges in $AH(\pi_1(M'))$. Notice further that if c is the core curve of any component of Z, then $\{l_{\rho_k \circ \phi'_k}(c)\}$ converges to a positive number $\delta(c)$. On the other hand, if c is the core curve of a component of X, then $\{l_{\rho_k \circ \phi'_k}(c)\}$ converges to 0.

In order to complete the proof, it is necessary to precompose by an additional sequence of automorphisms (and pass to an additional subsequence) to guarantee that the new sequence of representations converges on every component of $M - X$ which is not a thickened torus. We will make repeated use of the following elementary lemma, which we state without proof.

Lemma 5.7. *Let Q be a compact, hyperbolizable 3-manifold and let A be an essential annulus in Q with core curve a. Suppose that $\{\phi_i\}$ is a sequence in $AH(\pi_1(Q))$ such that if Q' is any component of $Q - A$, then $\{\phi_i|_{\pi_1(Q')}\}$ converges in $AH(\pi_1(Q'))$ and such that $\{l_{\phi_i}(a)\}$ converges to a positive number δ. Then, there exists a subsequence $\{\phi_k\}$ of $\{\phi_i\}$ and*

a sequence $\{g_k\}$ *of homeomorphisms of* Q *(each of which is a power of a Dehn twist about the annulus* A*) such that* $\{\phi_k \circ (g_k)_*\}$ *converges in* $AH(\pi_1(Q))$.

Let $Z_1, \ldots, Z_n$ be the components of Z. We inductively assume that we have chosen a subsequence $\{\rho_j\}$ of $\{\rho_k\}$ and a sequence $\{\phi_j\}$ of automorphisms of $\pi_1(M)$ such that if M' is a component of $M - (X \cup Z_l \cup \cdots \cup Z_n)$ which is not a thickened torus, then $\{\rho_j \circ \phi_j|_{\pi_1(M')}\}$ converges in $AH(\pi_1(M'))$. (If $l = 1$, then $\{\rho_j\}$ is the sequence $\{\rho_k\}$ obtained above and $\{\phi_j\} = \{\phi'_k\}$.)

Let Q_l be the component of $M - (X \cup Z_{l+1} \cup \cdots \cup Z_n)$ which contains Z_l. Then $\{\rho_j \circ \phi_j|_{\pi_1(Q_l)}\}$ and Z_l satisfy the hypotheses of Lemma 5.7. We may thus assume, perhaps after passing to a further subsequence, that there is a sequence $\{g_j\}$ of homeomorphisms of Q_l such that $\{\rho_j \circ \phi_j \circ (g_j)_*\}$ converges in $AH(\pi_1(Q_l))$. Since each g_j is a power of a Dehn twist about the annulus Z_l, we may extend $\{g_j\}$ to a sequence, again called $\{g_j\}$, of homeomorphisms of M, such that each g_j is the identity map on $M - Q_l$. Then, after replacing ϕ_j by $\phi_j \circ (g_j)_*$, we see that if M' is any component of $M - (X \cup Z_{l+1} \cup \cdots \cup Z_n)$ which is not a thickened torus, then $\{\rho_j \circ \phi_j|_{\pi_1(M')}\}$ converges in $AH(\pi_1(M'))$.

Applying the above process n times gives the desired subsequence and sequence of automorphisms. This completes the proof of Theorem 5.5. $\square$

6. 3-manifolds which are not generalized books of I-bundles

The main result of this section asserts that hyperbolizable 3-manifolds with incompressible boundary which are not generalized books of I-bundles have Λ-invariant strictly less than 1.

Theorem 2.9. *If* M *is a compact, orientable, hyperbolizable 3-manifold with incompressible boundary which is not a generalized book of* I*-bundles, then* $\Lambda(M) < 1$.

Proof of Theorem 2.9. Let M be a compact, orientable, hyperbolizable 3-manifold with incompressible boundary which is not a generalized book of I-bundles. We will assume that $\Lambda(M) = 1$ and arrive at a contradiction.

If $\Lambda(M) = 1$, there exists a sequence $\{N_i\}$ in $TT(M)$ such that $\{\lambda_0(N_i)\}$ converges to 1 where $N_i = \mathbf{H}^3/\Gamma_i$. Let $\rho_i : \pi_1(M) \to \mathrm{PSL}_2(\mathbf{C})$ be a discrete faithful representation with image Γ_i. Let $\{\rho_j\}$, $\{\phi_j\}$, X, and x be as in Theorem 5.5.

Let M_0 be a component of $M - X$ which contains a component V of $M - \Sigma(M)$ whose closure is not a solid torus or a thickened torus. Note that the fundamental group of V must be non-abelian, since any compact hyperbolizable 3-manifold with abelian fundamental group is a solid torus or a thickened torus. Hence, there exists a curve in V which is not homotopic into $\Sigma(M)$ and the closure of M_0 cannot be a thickened torus. Thus, Theorem 5.5 implies that $\{\rho_j \circ \phi_j|_{\pi_1(M_0)}\}$ converges to a discrete faithful representation $\rho : \pi_1(M_0) \to \mathrm{PSL}_2(\mathbf{C})$ such that $\rho(\pi_1(M_0 \cap X))$ is parabolic. Let $\Gamma^0 = \rho(\pi_1(M_0))$ and $N^0 = \mathbf{H}^3/\Gamma^0$. Let $\hat{\Gamma}$ be a geometric limit of some subsequence of $\{\Gamma^0_j = \rho_j(\phi_j(\pi_1(M_0)))\}$ and let $N^0_j = \mathbf{H}^3/\Gamma^0_j$ and $\hat{N} = \mathbf{H}^3/\hat{\Gamma}$. Since $1 \geq \lambda_0(N^0_j) \geq \lambda_0(N_j)$ and $\lim_{j\to\infty} \lambda_0(N_j) = 1$, we see that $\lim_{j\to\infty} \lambda_0(N^0_j) = 1$. Lemma 5.2 then implies that $\lambda_0(\hat{N}) = 1$ and hence that $\lambda_0(N^0) = 1$.

Let M_0' denote $M_0 - \mathcal{N}(X)$ where $\mathcal{N}(X)$ is a regular neighborhood of X and let Y denote the intersection of the closure of $\mathcal{N}(X)$ with M_0'. Since $\partial M_0' - Y$ is incompressible and the elements of Γ^0 corresponding to $\pi_1(Y)$ are parabolic, Lemma 5.1 implies that Γ^0 satisfies Bonahon's condition (B). Therefore, N^0 is topologically tame. Since, $\lambda_0(N^0) = 1$, the results of [13] imply that Γ^0 is either a Fuchsian group or an extended Fuchsian group. Thus N^0 contains a compact submanifold R which is a strong deformation retract of N^0 and is an I-bundle with base surface B, such that an element of Γ^0 is parabolic if and only if it is conjugate to an element of $\pi_1(\partial B)$. (If Z is the totally geodesic hyperplane preserved by Γ^0, then we can take B to be a compact core for Z/Γ^0 and R to be a regular neighborhood of B.)

Since $\Gamma^0 = \rho(\pi_1(M_0))$ and every element of Γ^0 corresponding to an element of $\pi_1(Y)$ is parabolic, there is a homotopy equivalence $h : M_0' \to R$ such that every element of Y maps into a component of S the sub-bundle over ∂B. If we let S_0 denote the set of components of S which contain images of elements of Y, then $h : (M_0', Y) \to (R, S_0)$ is a homotopy equivalence of pairs. Since every component of $\partial M_0' - Y$ is incompressible and h is a homotopy equivalence of pairs, every component of $\partial R - S_0$ is incompressible (see Proposition 1.2 in [6] or section 2 of Canary-McCullough [12]), which implies that $S_0 = S$. Thus, (M_0', Y) is homotopy equivalent to the I-pair (R, S) which implies that (M_0', Y) is homeomorphic to (R, S) (see corollary 5.8 in Johannson [18]). This implies that M_0' is an admissibly embedded essential I-bundle and hence may be properly homotoped into $\Sigma(M)$. This however contradicts the fact that there exist curves in V which are not homotopic into $\Sigma(M)$. This contradiction completes the proof of Theorem 2.9. $\square$

7. Remarks and Conjectures

1. The most natural analogue of the Gromov norm is the invariant $V(M)$ which is defined to be the infimum of the volumes of the convex cores of hyperbolic manifolds homeomorphic to the interior of M. One would make the following conjecture in the spirit of the paper.

Conjecture: $V(M) = 0$ *if and only if every component of* M*'s incompressible core is a generalized book of* I*-bundles.*

It seems likely that an argument similar to that in section 4 would give that $V(M) = 0$ whenever M is a generalized book of I-bundles. Whereas an argument similar to that in section 6, along with work of Taylor [31], should give that $V(M) > 0$ if M has incompressible boundary but is not a generalized book of I-bundles. If M is obtained from generalized books of I-bundles by adding 1-handles, then one needs to show that $V(M) = 0$. However, as the direct analogues for volumes of the results of Patterson for λ_0, used in Section 3, are false, one would need to find a more explicit proof for this case.

2. The work of Canary [11] and Burger-Canary [9] exhibits relationships between $V(M)$ and $\Lambda(M)$. The work of [11] gives the following upper bound for all M,

$$\Lambda(M) \leq \frac{4\pi|\chi(\partial M)|}{V(M)}$$

(where $\chi(\partial M)$ denotes the Euler characteristic of ∂M), while the work of [9] shows that there exist constants $A > 0$ and $B > 0$ such that if M has incompressible boundary then

$$\Lambda(M) \geq \frac{A}{(V(M) + B|\chi(\partial M)|)^2}.$$

3. If M is acylindrical then there is a unique hyperbolic manifold N whose convex core has totally geodesic boundary and is homeomorphic to M. One expects that $D(M) = d(N)$, $\Lambda(M) = \lambda_0(N)$, and that $V(M)$ is the volume $vol(C(N))$ of the convex core $C(N)$ of N. In fact, Bonahon [7] has shown that N is a local minimum for the volume (of the convex core) function on the space $GF(M)$ of geometrically finite hyperbolic 3-manifolds homeomorphic to the interior of M. More generally, if M has incompressible boundary and M_i is a component of $M - \Sigma(M)$ which is not homeomorphic to a solid torus, then there exists a unique hyperbolic 3-manifold N_i such that the convex core $C(N_i)$ has totally geodesic boundary and $C(N_i)$ is homeomorphic to M_i.

Conjecture: *If M has incompressible boundary, $\{M_1, \dots, M_n\}$ are the components of $M - \Sigma(M)$ which aren't homeomorphic to solid tori, and N_i are as above, then $V(M) = \sum_{i=1}^n vol(C(N_i))$ and $\Lambda(M) = min\{\lambda_0(N_i)\}$.*

A positive solution to the conjecture above would imply that Λ is a homotopy invariant, since the incompressible cores of homotopy-equivalent hyperbolic 3-manifolds are homotopy-equivalent, and Johannson's theorem [18] implies that the complements of the characteristic submanifolds of homotopy-equivalent hyperbolizable 3-manifolds with incompressible boundary are homeomorphic.

We note that $\sum_{i=1}^n vol(C(N_i))$ is equal to half the Gromov norm of the double of M. One can generalize this conjecture to obtain a similar conjecture for arbitrary compact hyperbolizable 3-manifolds.

4. It would be interesting to know more about the distribution of the set of values assumed by the invariant Λ. In this remark we show that the set of values accumulates at 0.

Let M_i be a compact hyperbolizable 3-manifold whose boundary is incompressible and has i toroidal boundary components and two genus two boundary components. One may obtain such manifolds by removing suitably chosen collections of boundary-parallel curves from a product $S \times [0,1]$ where S is a surface of genus 2. We will show that $V(M_i) \to \infty$, by observing that all but a bounded number of the cusps contribute a definite amount to the volume of the convex core of *any* hyperbolic structure on M_i.

If $x \in N$, then $inj_N(x)$ denotes the injectivity radius of the N at the point x. The Margulis lemma asserts that there exists a constant $\mathcal{M}_3$ such that if $\epsilon < \mathcal{M}_3$ and N is a complete hyperbolic 3-manifold, then every non-compact component of $N_{thin(\epsilon)} = \{x \in N | inj_N(x) < \epsilon\}$ is the quotient of a horoball by a group of parabolic transformations fixing the horoball. Moreover, for all $\mathcal{M}_3 > \epsilon > 0$, there exists $D(\epsilon) > 0$ such that any non-compact component of $N_{thin(\epsilon)}$ has volume at least $D(\epsilon)$. There also exists $K(\epsilon) > 0$ such that at most $K(\epsilon)|\chi(\partial C(N_i))|$ components of $N_{thin(\epsilon)}$ intersect $\partial C(N)$. If $N_i \in TT(M_i)$ then there are at least i non-compact components of $(N_i)_{thin(\epsilon)}$ (one for each toroidal component of ∂M_i.) Since $|\chi(\partial C(N_i))| \leq 4$ and every component of $(N_i)_{thin(\epsilon)}$ intersects $C(N_i)$, it follows that

$$vol(C(N_i)) > (i - 4K(\epsilon))D(\epsilon).$$

Hence, $V(M_i) \geq (i - 4K(\epsilon))D(\epsilon)$, so $V(M_i)$ converges to infinity and $\Lambda(M_i) \leq \frac{16\pi}{V(M_i)}$ converges to 0. On the other hand, $\Lambda(M_i) \neq 0$, by Proposition 2.5. Thus 0 is an accumulation point of the set of Λ values.

5. If Γ is a quasi-Fuchsian group, denote by $K(\Gamma)$ the minimal K for which there exists a K-quasiconformal map conjugating Γ to a Fuchsian group. One may use the methods of section 4 to construct a sequence $\{\Gamma_j\}$ of quasiconformally conjugate quasi-Fuchsian groups such that the Hausdorff dimensions of the limit sets of the Γ_j converge to 1, but $\{K(\Gamma_j)\}$ converges to ∞. We will discuss this more fully in a future note.

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